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DISTRIBUTIONS ON PARTITIONS, POINT PROCESSES, AND THE HYPERGEOMETRIC KERNEL

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Abstract. We study a 3–parametric family of stochastic point processes on the one–dimensional lattice originated from a remarkable family of representations of the infinite symmetric group. We prove that the correlation functions of the processes are given by determinantal formulas with a certain kernel. The kernel can be expressed through the Gauss hypergeometric function; we call it the hypergeometric kernel.

In a scaling limit our processes approximate the processes describing the decomposition of representations mentioned above into irreducibles. As we showed before, see math.RT/9810015, the correlation functions of these limit processes also have determinantal form with so–called Whittaker kernel. We show that the scaling limit of the hypergeometric kernel is the Whittaker kernel.

The integral operator corresponding to the Whittaker kernel is an integrable operator as defined by Its, Izergin, Korepin, and Slavnov. We argue that the hypergeometric kernel can be considered as a kernel defining a ‘discrete integrable operator’.

We also show that the hypergeometric kernel degenerates for certain values of parameters to the Christoffel–Darboux kernel for Meixner orthogonal polynomials. This fact is parallel to the degeneration of the Whittaker kernel to the Christoffel–Darboux kernel for Laguerre polynomials.

Contents

§0. Introduction

Let $\mathbb{Y}_n$ be the set of Young diagrams with n boxes and $\mathbb{Y} = \mathbb{Y}_0 \sqcup \mathbb{Y}_1 \sqcup \mathbb{Y}_2 \sqcup \dots$ be the set of all Young diagrams. In this paper we study a remarkable family of probability distributions on $\mathbb{Y}_n$, $n = 0, 1, 2, \dots$.

The whole picture depends on 2 parameters z and z' which satisfy certain conditions, see §1. For each pair (z, z') we have a probability distribution on every $\mathbb{Y}_n$,

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$n = 0, 1, 2, \dots$, we denote it by $M^{(n)}_{z,z'}$. Its value on a Young diagram $\lambda \in \mathbb{Y}_n$ with Frobenius coordinates $(p_1, \dots, p_d \,|\, q_1, \dots, q_d)$ has the form

$$
\begin{gathered}
M^{(n)}_{z,z'}(\lambda) = \frac{n!}{(zz')_n}\,(zz')^d \\
\times \prod_{i=1}^{d} \frac{(z+1)_{p_i}(z'+1)_{p_i}(-z+1)_{q_i}(-z'+1)_{q_i}}{p_i!p_i!q_i!q_i!}\,\det{}^2\left[\frac{1}{p_i+q_j+1}\right],
\end{gathered}
\tag{0.1}
$$

where $(a)_k$ stands for $a(a+1)\cdots(a+k-1)$.

The distributions $M^{(n)}_{z,z'}$ have a representation–theoretic meaning. Let S_n be the symmetric group of degree n, $S(\infty)$ be the union of the groups S_n, and for $\lambda \in \mathbb{Y}_n$, let χ^λ denote the irreducible character of S_n corresponding to λ. According to [KOV], there exists a central positive definite function $\chi^{(z,z')}$ on $S(\infty)$ such that, for any n, its restriction to S_n is

$$
\chi^{(z,z')}_n = \sum_{|\lambda|=n} M^{(n)}_{z,z'}(\lambda)\frac{\chi^\lambda}{\chi^\lambda(e)}\,.
$$

Moreover, the unitary representation $T^{(z,z')}$ corresponding to $\chi^{(z,z')}$ admits a nice geometric description (at least for $z' = \bar z$), see [KOV]. This representation–theoretic aspect was the original motivation of our study but in the present paper we do not discuss it (see [P.I]).

Let us associate to a Young diagram $\lambda = (p_1, \dots, p_d \,|\, q_1, \dots, q_d)$ a set of $2d$ points in $\mathbb{Z}' = \mathbb{Z} + \frac12$ as follows:

$$
\lambda = (p_1, \dots, p_d \,|\, q_1, \dots, q_d) \;\mapsto\; \{p_1 + \tfrac12, \dots, p_d + \tfrac12, -q_1 - \tfrac12, \dots, -q_d - \tfrac12\}.
$$

Then every probability measure on $\mathbb{Y}_n$ provides a probability measure on the set of all point configurations in $\mathbb{Z}'$ with equal number of points in $\mathbb{Z}'_+ = \mathbb{Z}' \cap \mathbb{R}_+$ and $\mathbb{Z}'_- = \mathbb{Z}' \cap \mathbb{R}_-$ and such that the total sum of absolute values of coordinates is equal to n.

Next, having a distribution on each $\mathbb{Y}_n$, we can mix them using a distribution on the set $\{0, 1, 2, \dots\}$ of indices n, then we get a probability distribution on $\mathbb{Y}$.

Thus, we get a probability measure on the set of all point configurations in $\mathbb{Z}'$ with equal number of points in $\mathbb{Z}'_+$ and $\mathbb{Z}'_-$. According to standard terminology, we can say that we defined a point process on $\mathbb{Z}'$.

Following a certain analogy with statistical physics, one can call the resulting object of the mixing procedure the *grand canonical ensemble*, see [V].

For our special distributions (0.1) we choose the mixing distribution to be the negative binomial distribution

$$
\mathrm{Prob}\{n\} = (1-\xi)^t\,\frac{(t)_n}{n!}\,\xi^n, \quad t = zz', \tag{0.2}
$$

where $\xi \in (0,1)$ is an additional parameter. (This choice is explained by willing to remove the factor $\frac{n!}{(t)_n}$ from the RHS of (0.1).) We shall denote by $\mathcal{P}_{z,z',\xi}$ the point process on $\mathbb{Z}'$ thus obtained.

The main result of this paper is the explicit computation of the correlation functions of $\mathcal{P}_{z,z',\xi}$. It turns out that they are given by determinantal formulas

$$\rho_n(x_1,\dots,x_n)=\det[K(x_i,x_j)]_{i,j=1}^n$$

with a certain kernel $K(x,y)$ on $\mathbb{Z}'$. This kernel can be expressed through the Gauss hypergeometric function. We call it the *hypergeometric kernel*.

Due to the representation theoretic origin of our problem, the distributions $M^{(n)}_{z,z'}$ have a number of additional properties. In particular, as $n\to\infty$, they converge to a probability measure on a certain limit object called the 'Thoma simplex', see [KOO] and §5 below.[1]

In terms of point processes, this implies that after an appropriate scaling the point processes $\mathcal{P}_{z,z',\xi}$ will converge, as $\xi\nearrow 1$, to a certain point process on $\mathbb{R}^*=\mathbb{R}\setminus\{0\}$ derived from the limit measure on the Thoma simplex (we shall give a rigorous proof of this result in our next paper).

This limit process has been thoroughly studied in our previous papers [P.I] – [P.V][2]. Its correlation functions also have determinantal form with so–called Whittaker kernel, see [P.IV]. In §5 we show directly that the scaling limit of the hypergeometric kernel is the Whittaker kernel.

The integral operator defined by the Whittaker kernel belongs to the class of integrable operators as defined by Its, Izergin, Korepin, Slavnov [IIKS]. We show that the operator corresponding to the hypergeometric kernel can be considered as an example of a 'discrete integrable operator'.

A. Okounkov pointed out that important information can be obtained from consideration of another degeneration of the point process introduced above. Assume that $z,z'\to\infty$ and $\xi=\frac{\eta}{zz'}=\frac{\eta}{t}\to 0$ where $\eta>0$ is fixed. Then the mixing distribution (0.2) tends to the Poisson distribution with parameter η:

$$\operatorname{Prob}\{n\}\to e^{-\eta}\,\frac{\eta^n}{n!},$$

while $M^{(n)}_{zz'}$ tends to the *Plancherel distribution* on $\mathbb{Y}_n$:

$$M^{(n)}_{zz'}(\lambda)\to M^{(n)}_{\infty}(\lambda)=\frac{\dim^2\lambda}{n!}\,, \tag{0.3}$$

where $\dim\lambda=\chi^\lambda(e)$ is the dimension of the irreducible representation of the symmetric group S_n corresponding to λ. Thus, we get an explicit formula for the correlation functions of the process governed by the poissonized Plancherel distributions.[3]

This formula allows to prove certain important facts about Plancherel distributions, see [BOO]. In particular, we were able to prove the conjecture by Baik, Deift, and Johansson [BDJ1, BDJ2] that the asymptotic behavior of $\lambda_1,\lambda_2,\dots$

[1]This is a kind of dual object to the infinite symmetric group, see [T], [VK], [KV]. The limit measure is, actually, a spectral measure for the decomposition of the representation $T^{(z,z')}$ into irreducibles, see [KOV].

[2]A survey of the results is given in [BO1].

[3]The limit relation (0.3) was known since the invention of the distributions $M^{(n)}_{zz'}$, see [KOV], but up to now it was not used.

with respect to the Plancherel distribution is governed by the Airy kernel [TW] and, therefore, coincides with that of the largest eigenvalues of a matrix from the Gaussian Unitary Ensemble. (Another approach to this conjecture can be found in [O].)

The paper is organized as follows. In §1 we introduce our main object of interest – the point process $\mathcal{P}_{z,z',\xi}$. In §2 we recall some generalities on determinantal point processes. The computation of the correlation functions of $\mathcal{P}_{z,z',\xi}$ and the formulas for the hypergeometric kernel can be found in §3. In §4 we show that if one of the parameters z, z' is an integer, the hypergeometric kernel degenerates to the Christoffel–Darboux kernel for the Meixner orthogonal polynomials on $\mathbb{Z}_+$.[4] In §5 we discuss the scaling limit of the hypergeometric kernel. §6 explains the connection with integrable operators. §7 is an appendix, we give there proofs of certain identities involving the Gauss hypergeometric functions which are used in §3.

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§1. Distributions on partitions. The grand canonical ensemble

For $n = 1, 2, \dots$, let $\mathbb{Y}_n$ denote the set of partitions of n, which will be identified with Young diagrams with n boxes. We agree that $\mathbb{Y}_0$ consists of a single element — the zero partition or the empty diagram $\varnothing$.

Given $\lambda \in \mathbb{Y}_n$, we write $|\lambda| = n$ and denote by $d = d(\lambda)$ the number of diagonal boxes in λ. We shall use the Frobenius notation [Ma]

$$\lambda = (p_1, \dots, p_d \,|\, q_1, \dots, q_d).$$

Here $p_i = \lambda_i - i$ is the number of boxes in the ith row of λ on the right of the ith diagonal box; likewise, $q_i = \lambda'_i - i$ is the number of boxes in the ith column of λ below the ith diagonal box (λ' stands for the transposed diagram).

Note that

$$p_1 > \dots > p_d \geq 0, \qquad q_1 > \dots > q_d \geq 0, \qquad \sum_{i=1}^{d} (p_i + q_i + 1) = |\lambda|.$$

The numbers p_i, q_i are called the *Frobenius coordinates* of the diagram λ.

Throughout the paper we fix two complex parameters z, z' such that the numbers $(z)_k (z')_k$ and $(-z)_k (-z')_k$ are real and strictly positive for any $k = 1, 2, \dots$. Here and below

$$(a)_k = a(a+1)\dots(a+k-1), \qquad (a)_0 = 1,$$

denotes the Pochhammer symbol.

The above assumption on z, z' means that one of the following two conditions holds:

- either $z' = \bar{z}$ and $z \in \mathbb{C} \setminus \mathbb{Z}$

[4] After the present paper was completed we learned that the 'Meixner kernel' has also arisen in the recent work [J].

• or $z, z' \in \mathbb{R}$ and there exists $m \in \mathbb{Z}$ such that $m < z, z' < m+1$.

We set

$$t = zz'$$

and note that $t > 0$.

For a Young diagram λ let $\dim\lambda$ denote the number of the standard Young tableaux of shape λ. Equivalently, $\dim\lambda$ is the dimension of the irreducible representation (of the symmetric group of degree $|\lambda|$) corresponding to λ, see [Ma]. In the Frobenius notation,

$$\frac{\dim\lambda}{|\lambda|!} = \frac{\prod\limits_{1\le i<j\le d}(p_i-p_j)(q_i-q_j)}{\prod\limits_{1\le i,j\le n}(p_i+q_j+1)} = \det\left[\frac{1}{p_i+q_j+1}\right]_{1\le i,j\le d},$$

see, e.g., [P.I, Prop. 2.6].

We introduce a function on the Young diagrams depending on the parameters z, z':

$$M_{z,z'}(\lambda) = \frac{t^d}{(t)_n}\prod_{i=1}^{d}\frac{(z+1)_{p_i}(z'+1)_{p_i}(-z+1)_{q_i}(-z'+1)_{q_i}}{p_i!p_i!q_i!q_i!}\,\frac{\dim^2\lambda}{|\lambda|!}$$

$$= \frac{|\lambda|!}{(t)_n}\,t^d\prod_{i=1}^{d}\frac{(z+1)_{p_i}(z'+1)_{p_i}(-z+1)_{q_i}(-z'+1)_{q_i}}{p_i!p_i!q_i!q_i!}\det{}^2\left[\frac{1}{p_i+q_j+1}\right].$$

We agree that $M_{z,z'}(\varnothing) = 1$. Thanks to our assumption on the parameters, $M_{z,z'}(\lambda) > 0$ for all λ.

Proposition 1.1. *For any n,*

$$\sum_{\lambda\in\mathbb{Y}_n} M_{z,z'}(\lambda) = 1,$$

so that the restriction of $M_{z,z'}$ to $\mathbb{Y}_n$ is a probability distribution on $\mathbb{Y}_n$.

We shall denote this distribution by $M^{(n)}_{z,z'}$.

Comments. This result is the starting point of our investigations. About its origin and representation–theoretic significance, see [KOV]. Several direct proofs of the proposition are known. E.g., a simple proof is given in [Part I, §7]. About generalizations, see [K], [BO2]. Note that

$$\lim_{|z|,|z'|\to\infty} M_{z,z'}(\lambda) = \frac{\dim^2\lambda}{|\lambda|!},$$

so that the limit form of the identity of the proposition is

$$\sum_{\lambda\in\mathbb{Y}_n}\frac{\dim^2\lambda}{|\lambda|!} = 1,$$

which is well known.

Let $\mathbb{Y} = \mathbb{Y}_0 \sqcup \mathbb{Y}_1 \sqcup \dots$ denote the set of all Young diagrams. Consider the negative binomial distribution on the nonnegative integers, which depends on t and the additional parameter ξ, $0 < \xi < 1$:

$$\pi_{t,\xi}(n) = (1-\xi)^t \, \frac{(t)_n}{n!} \, \xi^n \,, \qquad n = 0, 1, \dots .$$

For $\lambda \in \mathbb{Y}$ we set

$$M_{z,z',\xi}(\lambda) = M_{z,z'}(\lambda)\, \pi_{t,\xi}(|\lambda|).$$

By the construction, $M_{z,z',\xi}(\cdot)$ is a probability distribution on $\mathbb{Y}$, which can be viewed as a mixing of the finite distributions $M^{(n)}_{z,z'}$. From the formulas for $M_{z,z'}$ and $\pi_{t,\xi}$ we get an explicit expression for $M_{z,z',\xi}$:

$$M_{z,z',\xi}(\lambda) = (1-\xi)^t \, \xi^{\sum\limits_{i=1}^{d}(p_i+q_i+1)} \, t^d \\ \times \prod_{i=1}^{d} \frac{(z+1)_{p_i}(z'+1)_{p_i}(-z+1)_{q_i}(-z'+1)_{q_i}}{p_i!p_i!q_i!q_i!} \det{}^2 \left[\frac{1}{p_i+q_j+1} \right] . \tag{1.1}$$

Following a certain analogy with models of statistical physics (cf. [V]) one may call $(\mathbb{Y}, M_{z,z',\xi})$ the *grand canonical ensemble.*

Let $\mathbb{Z}'$ denote the set of half–integers,

$$\mathbb{Z}' = \mathbb{Z} + \tfrac{1}{2} = \{ \dots, -\tfrac{3}{2}, -\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{3}{2}, \dots \},$$

and let $\mathbb{Z}'_+$ and $\mathbb{Z}'_-$ be the subsets of positive and negative half–integers, respectively. It will be sometimes convenient to identify both $\mathbb{Z}'_+$ and $\mathbb{Z}'_-$ with $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$ by making use of the correspondence $\pm(k+\frac{1}{2}) \leftrightarrow k$, where $k \in \mathbb{Z}_+$.

Denote by $\mathrm{Conf}(\mathbb{Z}')$ the space of all finite subsets of $\mathbb{Z}'$ which will be called *configurations*. We define an embedding $\lambda \mapsto X$ of the set $\mathbb{Y}$ of Young diagrams into the set $\mathrm{Conf}(\mathbb{Z}')$ of configurations in $\mathbb{Z}'$ as follows:

$$\lambda = (p_1, \dots, p_d \,|\, q_1, \dots, q_d) \;\mapsto\; X = \{ p_1 + \tfrac{1}{2}, \dots, p_d + \tfrac{1}{2}, -q_1 - \tfrac{1}{2}, \dots, -q_d - \tfrac{1}{2} \}.$$

Under the identification $\mathbb{Z}' \simeq \mathbb{Z}_+ \sqcup \mathbb{Z}_+$, the map $\lambda \mapsto X$ is simply associating to λ the collection of its Frobenius coordinates. The image of the map consists exactly of the configurations X with the property $|X \cap \mathbb{Z}'_+| = |X \cap \mathbb{Z}'_-|$.

Under the embedding $\lambda \mapsto X$ the probability measure $M_{z,z',\xi}$ on $\mathbb{Y}$ turns into a probability measure on the configurations in $\mathbb{Z}'$. Following the conventional terminology, see [DVJ], we get a point process on $\mathbb{Z}'$; let us denote it as $\mathcal{P}_{z,z',\xi}$.

Our primary goal is to compute the correlation functions of this point process.

§2. Determinantal point processes

Let $\mathfrak{X}$ be a countable set. Its finite subsets will be called *configurations* and denoted by the letters X, Y. The space of all configurations is denoted as $\mathrm{Conf}(\mathfrak{X})$; this is a discrete space. In this section, by a *point process* on $\mathfrak{X}$ we mean a map from a probability space to $\mathrm{Conf}(\mathfrak{X})$.[5] Let $\mathcal{P}$ be a point process on $\mathfrak{X}$. It induces a

[5]Actually, such a definition is rather restricted but it suffices for our purpose. For general axiomatics of point processes, see [DVJ], [L1], [L2].

probability distribution on $\mathrm{Conf}(\mathfrak{X})$, which is a nonnegative function $\pi(X)$ such that $\sum_X \pi(X) = 1$. One may simply identify $\mathcal{P}$ and π: then the underlying probability space is $\mathrm{Conf}(\mathfrak{X})$ itself.

We introduce a related function on $\mathrm{Conf}(\mathfrak{X})$ as follows:

$$\rho(X) = \sum_{Y \supseteq X} \pi(Y).$$

That is, $\rho(X)$ is the probability that the random configuration contains X.

Consider the Hilbert space $\ell^2(\mathfrak{X})$. An operator L in $\ell^2(\mathfrak{X})$ will be viewed as an infinite matrix $L(x, y)$ whose rows and columns are indexed by points of $\mathfrak{X}$. By L_X we denote the finite matrix of format $X \times X$ which is obtained from L by letting x, y range over X.

Assume L is a trace class operator in $\mathfrak{X}$ such that all the principal minors $\det L_X$ are real nonnegative numbers. We agree that $\det L_\varnothing = 1$. We have

$$\det(1 + L) = \sum_n \mathrm{tr}(\wedge^n L) = \sum_X \det L_X$$

(here $\wedge^n L$ stands for the nth exterior power of L acting in the nth exterior power of the Hilbert space $\ell^2(\mathfrak{X})$). Thus, we can define a point process by

$$\pi(X) = \frac{\det L_X}{\det(1 + L)}\,.$$

Let us call it the *determinantal point process* determined by the operator L.[6]

Proposition 2.1. *Let L satisfy the above assumption, $\pi(\cdot)$ be the corresponding point process, and $\rho(\cdot)$ be the associated function as defined above. Set $K = L(1 + L)^{-1}$. Then*

$$\rho(X) = \det K_X\,.$$

Proof. We shall reproduce the argument indicated in [DVJ]. Let $f(x)$ be a function on $\mathfrak{X}$ such that $f_0(x) = f(x) - 1$ is finitely supported. For any point process $\pi(\cdot)$,

$$\sum_Y \pi(Y) \prod_{y \in Y} f(y) = \sum_Y \pi(Y) \prod_{y \in Y} (1 + f_0(y)) = \sum_X \rho(X) \prod_{x \in X} f_0(x).$$

When the process is defined by an operator L then, identifying f with the diagonal matrix with diagonal entries $f(x)$, we get

$$\begin{aligned}
\sum_Y \pi(Y) \prod_{y \in Y} f(y) &= \left(\sum_Y \det L_Y \prod_{y \in Y} f(y) \right) \det{}^{-1}(1 + L) \\
&= \det(1 + fL)\det{}^{-1}(1 + L) = \det\big((1 + fL)(1 + L)^{-1}\big) \\
&= \det\big((1 + L + f_0 L)(1 + L)^{-1}\big) = \det(1 + f_0 K) = \sum_X \det K_X \prod_{x \in X} f_0(x)\,.
\end{aligned}$$

[6] This term is not a conventional one. Such processes, not necessarily on discrete spaces, arise in different topics, in particular, in connection with random matrices. In [DVJ], they are called ‘fermion processes’ but in the the random matrix literature no special term is adopted.

Thus,

$$\sum_X \rho(X) \prod_{x\in X} f_0(x) = \sum_X \det K_X \prod_{x\in X} f_0(x)$$

for any finitely supported function f_0, which implies $\rho(X) = \det K_X$. $\square$

Let ρ_n be the restriction of $\rho(\cdot)$ to the n-point configurations. One can view ρ_n as a symmetric function in n variables,

$$\rho_n(x_1,\dots,x_n) = \rho(\{x_1,\dots,x_n\}), \quad x_1,\dots,x_n \text{ pairwise distinct.}$$

In this notation, the result of Proposition 2.1 reads as follows

$$\rho_n(x_1,\dots,x_n) = \det[K(x_i,x_j)]_{1\le i,j\le n}\,.$$

We call ρ_n the *n-point correlation function.*

From now on we assume that $\mathfrak{X} = \mathfrak{X}^+ \sqcup \mathfrak{X}^-$ (disjoint union of two countable sets) and we write $\ell^2(\mathfrak{X}) = \ell^2(\mathfrak{X}^+) \oplus \ell^2(\mathfrak{X}^-)$. According to this decomposition we write operators in $\ell^2(\mathfrak{X})$ as 2×2 operator matrices,

$$L = \begin{bmatrix} L_{++} & L_{+-} \\ L_{-+} & L_{--} \end{bmatrix}, \qquad K = \begin{bmatrix} K_{++} & K_{+-} \\ K_{-+} & K_{--} \end{bmatrix}.$$

Given a configuration X, we set $X^\pm = X \cap \mathfrak{X}^\pm$. We shall deal with operators L such that $L_{++}=0$, $L_{--}=0$. Then, as is easily seen, $\pi(X)=0$ unless $|X^+| = |X^-|$.

Proposition 2.2. *The transforms $L \mapsto K = L(1+L)^{-1}$ and $K \mapsto L = K(1-K)^{-1}$ define a bijective correspondence between*

(i) *the operators L of the form*

$$L = \begin{bmatrix} 0 & A \\ -B & 0 \end{bmatrix}, \tag{2.1}$$

where the matrix $1+AB$ is invertible (equivalently, $1+BA$ is invertible) and

(ii) *the operators K of the form*

$$K = \begin{bmatrix} CD & C \\ DCD - D & DC \end{bmatrix}, \tag{2.2}$$

where $1-CD$ is invertible (equivalently, $1-DC$ is invertible).

In terms of the blocks, this correspondence takes the form

$$C = (1+AB)^{-1}A = A(1+BA)^{-1}, \quad D = B, \tag{2.3}$$

$$A = C(1-DC)^{-1} = (1-CD)^{-1}C, \quad B = D. \tag{2.4}$$

In particular,

$$1 - CD = (1+AB)^{-1}, \quad 1 - DC = (1+BA)^{-1}. \tag{2.5}$$

Proof. The proof is straightforward, see [P.V, Prop. 2.2]. $\square$

Proposition 2.3. *Let K be a J–Hermitian[7] kernel of the form* (2.2)*. Then*

$$L = \begin{bmatrix} 0 & D^* \\ -D & 0 \end{bmatrix}.$$

Proof. By Proposition 2.2, L is given by the formula (2.1) with $B=D$. Since K is J-Hermitian, L is J-Hermitian, too. This implies $A = B^* = D^*$. $\square$

[7] I.e., Hermitian with respect to the indefinite inner product determined by the matrix $J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

§3. Calculation of the correlation functions. The hypergeometric kernel

In this section we shall apply the formalism of §2 to the point processes $\mathcal{P}_{z,z',\xi}$ introduced at the end of §1.

Naturally, we specify $\mathfrak{X} = \mathbb{Z}'$ and $\mathfrak{X}^{\pm} = \mathbb{Z}'_{\pm}$.

Let us introduce 2 meromoprhic functions in u depending on z, z', ξ as parameters

$$\begin{aligned}\psi_{\pm}(u) &= t^{1/2}\,\xi^{u+1/2}\,(1-\xi)^{\pm(z+z')}\,\frac{\Gamma(u+1\pm z)\Gamma(u+1\pm z')}{\Gamma(1\pm z)\Gamma(1\pm z')\Gamma(u+1)\Gamma(u+1)}\\ &= t^{-1/2}\,\xi^{u+1/2}\,(1-\xi)^{\pm(z+z')}\,\frac{\Gamma(u+1\pm z)\Gamma(u+1\pm z')}{\Gamma(\pm z)\Gamma(\pm z')\Gamma(u+1)\Gamma(u+1)}.\end{aligned} \tag{3.1}$$

An important fact is that the functions $\psi_{\pm}(u)$ have exponential decay as u tends to $+\infty$ along the real axis. (Indeed, since $\xi \in (0,1)$, the factor ξ^u has exponential decay; as for the remaining expression in (3.1), it behaves as a constant times $u^{\pm(z+z')}$, so that it has at most polynomial growth.)

We shall consider two diagonal matrices of format $\mathbb{Z}_+ \times \mathbb{Z}_+$, depending on the parameters z, z', ξ and denoted as $\Psi_{\pm}$, and a third matrix of the same format denoted as W:

$$\Psi_{\pm} = \operatorname{diag}\left\{\frac{t^{1/2}\xi^{k+1/2}(1-\xi)^{\pm(z+z')}(1\pm z)_k(1\pm z')_k}{k!\,k!}\right\}, \quad W(k,l) = \frac{1}{k+l+1}, \tag{3.2}$$

where k, l range over $\mathbb{Z}_+$. Note that the kth diagonal entry of $\Psi_{\pm}$ equals $\psi_{\pm}(k)$. As the diagonal entries of $\Psi_{\pm}$ are real and positive, we may introduce the diagonal matrices $\Psi_{\pm}^{1/2}$ which are real and positive, too. Note also that W is real and symmetric.

Proposition 3.1. *The point process $\mathcal{P}_{z,z',\xi}$ is a determinantal process in the sense of §2, and the corresponding operator L is as follows:*

$$L = \begin{bmatrix} 0 & \Psi_+^{1/2} W \Psi_-^{1/2} \\ -\Psi_-^{1/2} W \Psi_+^{1/2} & 0 \end{bmatrix}. \tag{3.3}$$

Note that L is real and J–symmetric.

Proof. Let λ be a Young diagram and X be the corresponding configuration. We must prove that

$$M_{z,z',\xi}(\lambda) = \frac{\det L_X}{\det(1+L)}. \tag{3.4}$$

Since L has the form $\begin{bmatrix} 0 & A \\ -A' & 0 \end{bmatrix}$, where prime means transposition, we have, taking account of the exact expression for the matrix A (see (3.3)):

$$\det L_X = \det{}^2[A(p_i,q_j)]_{1\le i,j\le d} = \prod_{i=1}^{d} \psi_+(p_i)\psi_-(q_i) \cdot \det{}^2\left[\frac{1}{p_i+q_j+1}\right].$$

In the latter product, the factor $(1-\xi)^{z+z'}$ coming from ψ_+ cancels with the factor $(1-\xi)^{-(z+z')}$, and we get

$$\det L_X = \det(1+L)^{-1}\, \xi^{\sum\limits_{i=1}^{d}(p_i+q_i+1)}\, t^d$$
$$\times \prod_{i=1}^{d} \frac{(z+1)_{p_i}(z'+1)_{p_i}(-z+1)_{q_i}(-z'+1)_{q_i}}{p_i!p_i!q_i!q_i!} \det{}^2\left[\frac{1}{p_i+q_j+1}\right].$$

Comparing this with the expression (1.1) for $M_{z,z',\xi}(\lambda)$, we see that they coincide up to a constant factor which does not depend on λ. Since the both expressions define probability distributions, we conclude that they are identical. □

Remark 3.2. As a by–product of the proof we get the following result:

$$\det(1+L) = \det(1+\Psi_+^{1/2} W \Psi_- W \Psi_+^{1/2}) = (1-\xi)^{-t}.$$

By Propositions 3.1 and 2.1, the correlation functions of the point process $\mathcal{P}_{z,z',\xi}$ are given by the determinantal formula involving the operator $K = L(1+L)^{-1}$. Theorem 3.3 below provides an explicit expression for this operator.

Introduce the functions

$$R_\pm(u) = \psi_\pm(u)\, F(\mp z, \mp z'; u+1; \tfrac{\xi}{\xi-1}), \tag{3.5}$$

$$S_\pm(u) = \frac{t^{1/2}\xi^{1/2}\,\psi_\pm(u)}{1-\xi}\, \frac{F(1\mp z, 1\mp z'; u+2; \tfrac{\xi}{\xi-1})}{u+1}, \tag{3.6}$$

$$P_\pm(u) = (\psi_\pm(u))^{-1/2} R_\pm(u), \quad Q_\pm(u) = (\psi_\pm(u))^{-1/2} S_\pm(u). \tag{3.7}$$

They are all well–defined for $u \ge 0$, because $\psi_\pm(u)$ is strictly positive for $u \ge 0$. In particular, they are well–defined at the points $u = k \in \mathbb{Z}_+$.

Note that the hypergeometric function that enters (3.5) or (3.6) remains bounded as $u \to +\infty$. Hence, the exponential decay of $\psi_\pm(u)$ implies the exponential decay of $R_\pm(u)$ and $S_\pm(u)$ as u tends to $+\infty$ along the real axis.

Theorem 3.3. *Let* $K = \begin{bmatrix} K_{++} & K_{+-} \\ K_{-+} & K_{--} \end{bmatrix}$ *be the operator in* $\ell^2(\mathbb{Z}_+) \oplus \ell^2(\mathbb{Z}_+)$ *with the blocks*

$$K_{++}(k,l) = \frac{P_+(k)Q_+(l) - Q_+(k)P_+(l)}{k-l},$$

$$K_{+-}(k,l) = \frac{P_+(k)P_-(l) + Q_+(k)Q_-(l)}{k+l+1},$$

$$K_{-+}(k,l) = -\frac{P_-(k)P_+(l) + Q_-(k)Q_+(l)}{k+l+1},$$

$$K_{--}(k,l) = \frac{P_-(k)Q_-(l) - Q_-(k)P_-(l)}{k-l}.$$

Here the functions $P_\pm(u)$ *and* $Q_\pm(u)$ *are defined in* (3.7), *and the expressions* $K_\pm(k,l)\big|_{k=l}$ *are understood according to the L'Hospital rule.*

Then $K = L(1+L)^{-1}$, where L is defined in (3.3).

We shall call the kernel K defined above the *hypergeometric kernel.*

Note that K is J–symmetric (because of the minus sign in the expression for K_{-+}).

Proof. We shall prove that K has the form (2.2) with

$$C = K_{+-}\,, \qquad D = -L_{-+} = \Psi_-^{1/2} W \Psi_+^{1/2}\,. \tag{3.8}$$

I.e.,

$$K_{++} = CD, \qquad K_{--} = DC, \qquad K_{-+} = DCD - D. \tag{3.9}$$

As K is J–symmetric, the desired result will follow from Proposition 2.3.[8]

It is convenient to slightly rewrite the desired relations (3.9) in order to avoid square roots. To do this, we set

$$N = \Psi_+^{1/2} C \Psi_-^{1/2} = \Psi_+ W \Psi_-\,.$$

I.e.,

$$N(k,l) = \frac{R_+(k)R_-(l) + S_+(k)S_-(l)}{k+l+1}\,.$$

By virtue of the connection between $P_\pm, Q_\pm$ and $R_\pm, S_\pm$, the relations (3.9) are equivalent to the following ones:

$$(NW)(k,l) = \frac{1}{\psi_+(l)}\,\frac{R_+(k)S_+(l) - S_+(k)R_+(l)}{k-l}\,, \tag{3.10}$$

$$(WN)(k,l) = \frac{1}{\psi_-(k)}\,\frac{R_-(k)S_-(l) - S_-(k)R_-(l)}{k-l}\,, \tag{3.11}$$

$$(WNW - W)(k,l) = \frac{1}{\psi_-(k)\psi_+(l)}\,\frac{R_-(k)R_+(l) - S_-(k)S_+(l)}{k+l+1}\,. \tag{3.12}$$

We note once more that, by agreement, the indeterminacy arising in (3.10) and (3.11) for $k = l$ is removed by making use of the L'Hospital rule.

To prove the relations above we shall need certain identities involving the hypergeometric function.

Lemma 3.4. *Set*

$$\widehat{R}_\pm(u) = \sum_{k=0}^{\infty} \frac{R_\pm(k)}{u+k+1}\,, \qquad \widehat{S}_\pm(u) = \sum_{k=0}^{\infty} \frac{S_\pm(k)}{u+k+1}\,. \tag{3.13}$$

Then the series absolutely converge for $u \neq -1, -2, \dots$ and the following relations hold

$$\widehat{R}_\pm(u) = \psi_\mp(u)^{-1}\, S_\mp(u), \qquad \widehat{S}_\pm(u) = 1 - \psi_\mp(u)^{-1}\, R_\mp(u). \tag{3.14}$$

[8] In the same way, one could verify directly that the operators C, D obey the relations (2.3) which means that K coincides with $L(1+L)^{-1}$. However, reference to Proposition 2.3 makes this verification redundant.

Lemma 3.5.

$$R_+(u)R_-(-u-1) + S_+(u)S_-(-u-1) = \psi_+(u)\psi_-(-u-1). \tag{3.15}$$

The proofs of these two lemmas can be found in the Appendix (§7).

Let us now check (3.10). By the definition of N and W,

$$(NW)(k,l) = \sum_{j=0}^{\infty} \frac{R_+(k)R_-(j) + S_+(k)S_-(j)}{(k+j+1)(j+l+1)}\,. \tag{3.16}$$

Assume first $k \neq l$. We write

$$\frac{1}{(k+j+1)(j+l+1)} = \frac{1}{k-l}\left(\frac{1}{l+j+1} - \frac{1}{k+j+1}\right), \tag{3.17}$$

and plug this into (3.16). Then we get

$$\begin{aligned}(NW)(k,l) = \frac{R_+(k)}{k-l}\sum_{j=0}^{\infty}\frac{R_-(j)}{l+j+1} + \frac{S_+(k)}{k-l}\sum_{j=0}^{\infty}\frac{S_-(j)}{l+j+1} \\ -\frac{R_+(k)}{k-l}\sum_{j=0}^{\infty}\frac{R_-(j)}{k+j+1} - \frac{S_+(k)}{k-l}\sum_{j=0}^{\infty}\frac{S_-(j)}{k+j+1}\,.\end{aligned}$$

By (3.13), this can be written as

$$(NW)(k,l) = \frac{R_+(k)\widehat{R}_-(l) + S_+(k)\widehat{S}_-(l)}{k-l} - \frac{R_+(k)\widehat{R}_-(k) + S_+(k)\widehat{S}_-(k)}{k-l}\,.$$

Applying (3.14), we get

$$\begin{aligned}R_+(k)\widehat{R}_-(l) + S_+(k)\widehat{S}_-(l) &= \frac{R_+(k)S_+(l) - S_+(k)R_+(l)}{\psi_+(l)}\,, \\ R_+(k)\widehat{R}_-(k) + S_+(k)\widehat{S}_-(k) &= \frac{R_+(k)S_+(k) - S_+(k)R_+(k)}{\psi_+(k)} = 0.\end{aligned}$$

Thus, we have checked (3.10) for $k \neq l$.

To extend the argument above to the case $k = l$, we replace (3.17) by a slightly more complicated expression that makes sense for any $k, l \in \mathbb{Z}_+$:

$$\frac{1}{(k+j+1)(j+l+1)} = \lim_{u\to l}\left\{\frac{1}{k-u}\left(\frac{1}{u+j+1} - \frac{1}{k+j+1}\right)\right\}, \tag{3.18}$$

where u is assumed to be nonintegral. Since the functions $R_\pm(u)$, $S_\pm(u)$ have exponential decay as $u \to +\infty$ (see the paragraph before Theorem 3.3), we may interchange summation and the limit transition. Then we can repeat all the transformations. At the very end we must pass to the limit as $u \to l$, which means that we follow the L'Hospital rule. This concludes the proof of (3.16).

The proof of (3.11) is quite similar, and we proceed to the proof of (3.12).

By virtue of the expression (3.10) for NW and our agreement about the L'Hospital rule, we get

$$(WNW)(k,l) = \lim_{u\to l} \left\{ \sum_{j=0}^{\infty} \frac{R_+(j)S_+(u) - S_+(j)R_+(u)}{\psi_+(u)(k+j+1)(j-u)} \right\}. \tag{3.19}$$

Using the transformation

$$\frac{1}{(k+j+1)(j-u)} = -\frac{1}{k+u+1}\left(\frac{1}{k+j+1} - \frac{1}{(-u-1)+j+1}\right)$$

we rewrite the above sum as follows:

$$\begin{aligned}
&\sum_{j=0}^{\infty} \frac{R_+(j)S_+(u) - S_+(j)R_+(u)}{\psi_+(u)(k+j+1)(j-u)} \\
&\quad = -\frac{\widehat{R}_+(k)S_+(u) - \widehat{S}_+(k)R_+(u)}{k+u+1} + \frac{\widehat{R}_+(-u-1)S_+(u) - \widehat{S}_+(-u-1)R_+(u)}{k+u+1}.
\end{aligned}$$

Next, applying (3.14), we transform this to

$$\begin{aligned}
&-\frac{S_-(k)S_+(u) + R_-(k)R_+(u)}{\psi_-(k)(k+u+1)} - \frac{R_+(u)}{k+u+1} \\
&\qquad + \frac{S_-(-u-1)S_+(u) + R_-(-u-1)R_+(u)}{\psi_-(-u-1)(k+u+1)} + \frac{R_+(u)}{k+u+1}.
\end{aligned}$$

Here the second and the fourth fractions cancel each other, while the third fraction equals $\psi_+(u)/(k+u+1)$, because of (3.15). Consequently, the whole expression is equal to

$$-\frac{S_-(k)S_+(u) + R_-(k)R_+(u)}{\psi_-(k)(k+u+1)} + \frac{\psi_+(u)}{k+u+1}.$$

Substituting this expression instead of the sum in (3.19), we get

$$\begin{aligned}
(WNW)(k,l) &= \lim_{u\to l}\left\{-\frac{S_-(k)S_+(u) + R_-(k)R_+(u)}{\psi_-(k)\psi_+(u)(k+u+1)} + \frac{1}{k+u+1}\right\} \\
&= -\frac{S_-(k)S_+(l) + R_-(k)R_+(l)}{\psi_-(k)\psi_+(l)(k+l+1)} + \frac{1}{k+l+1}.
\end{aligned}$$

Thus,

$$(WNW - W)(k,l) = -\frac{S_-(k)S_+(l) + R_-(k)R_+(l)}{\psi_-(k)\psi_+(l)(k+l+1)}$$

which proves (3.12).

This completes the proof of the theorem. $\square$

§4 Connection with Meixner polynomials

In this section we shall show that when one of the parameters $z,\ z'$ becomes an integer, the '++'–block of the hypergeometric kernel defined in Theorem 3.3 turns into the Christoffel–Darboux kernel for Meixner orthogonal polynomials.

The Meixner polynomials form a system $\{\mathfrak{M}_n(k;\alpha+1,\xi)\}$ of orthogonal polynomials, which corresponds to the following weight function on $\mathbb{Z}_+$:

$$f(k)=f_{\alpha,\xi}(k)=\frac{(\alpha+1)_k\xi^k}{k!}=\frac{\Gamma(\alpha+1+k)\xi^k}{\Gamma(\alpha+1)k!}\,,\qquad k\in\mathbb{Z}_+\,.$$

Here $k\in\mathbb{Z}_+$ is the argument and $\alpha>-1$ and $0<\xi<1$ are parameters; $\deg\mathfrak{M}_n(k;\alpha+1,\xi)=n$. For a detailed information about these polynomials see [NSU], [KS].[9]

Meixner polynomials can be expressed through the Gauss hypergeometric function:

$$\mathfrak{M}_n(k;\alpha+1,\xi)=F(-n,-k;\alpha+1;\tfrac{\xi-1}{\xi})$$
$$=\frac{k!\Gamma(-\alpha-n)}{\Gamma(1+k-n)\Gamma(-\alpha)}\left(\frac{1-\xi}{\xi}\right)^n F(-n,-\alpha-n;1+k-n;\tfrac{\xi}{\xi-1}).$$

Basic constants related to these polynomials have the form

$$\mathfrak{M}_n(k;\alpha+1,\xi)=a_nk^n+\{\text{lower degree terms in }k\},\quad a_n=\left(\frac{1-\xi}{\xi}\right)^n\frac{1}{(\alpha+1)_n}$$
$$h_n=||\mathfrak{M}_n(k;\alpha+1,\xi)||^2=\sum_{k=0}^{\infty}\mathfrak{M}_n^2(k;\alpha+1,\xi)f(k)=\frac{n!}{\xi^n(1-\xi)^{\alpha+1}(\alpha+1)_n}\,.$$

Consider the Nth Christoffel–Darboux kernel for the Meixner polynomials. It projects the Hilbert space $\ell^2(\mathbb{Z}_+,f(\cdot))$ on the N–dimensional subspace spanned by the polynomials of degree $\le N-1$. Let us pass from $\ell^2(\mathbb{Z}_+,f(\cdot))$ to the ordinary ℓ^2 space on $\mathbb{Z}_+$, which corresponds to the counting measure. Then the Christoffel–Darboux kernel will be transformed to a certain kernel, which will be called the *Meixner kernel* and denoted as $M_N(k,l)$. We have:

$$M_N(k,l)=\sum_{n=0}^{N-1}\frac{\mathfrak{M}_n(k;\alpha+1,\xi)\,\mathfrak{M}_n(l;\alpha+1,\xi)}{h_n}\sqrt{f(k)f(l)}=\frac{a_{N-1}}{a_Nh_{N-1}}\sqrt{f(k)f(l)}$$
$$\times\frac{\mathfrak{M}_N(k;\alpha+1,\xi)\mathfrak{M}_{N-1}(l;\alpha+1,\xi)-\mathfrak{M}_{N-1}(k;\alpha+1,\xi)\mathfrak{M}_N(l;\alpha+1,\xi)}{k-l}\,.$$

Proposition 4.1. *Let $z=N+\alpha$, $z'=N$, and let $K_{++}(k,l)$ be the "++" block of the corresponding hypergeometric kernel. Then*

$$K_{++}(k,l)=M_N(k+N,l+N).$$

[9] Our normalization of the Meixner polynomials coincides with that of [KS] and slightly differs from that of [NSU].

Proof. The proof is straightforward. □

Consider the N–point "Meixner ensemble" on $\mathbb{Z}_+$ whose joint probability distribution has the form

$$p(k_1,\dots,k_N)=const\cdot\prod_{1\le i<j\le N}(k_i-k_j)^2\prod_{i=1}^{n}f_{\alpha,\xi}(k_i).$$

The standard argument due to Dyson (see [Dy], [Me]) shows that the correlation functions of this ensemble are given by determinantal formulas with the Meixner kernel:

$$\rho_n(x_i,\dots,x_n)=\det[M_N(k_i,k_j)]_{i,j=1}^{n}.$$

Then Proposition 4.1 shows that our point process $\mathcal{P}_{z,z',\xi}$ restricted to the positive copy of $\mathbb{Z}_+$ for $z=N+\alpha$, $z'=N$ coincides with the trace of the N–point Meixner ensemble on the set $\{N+1,N+2,\dots\}$. In this subset the number of points of the Meixner ensemble can vary from 0 to N, which agrees with our picture.

§5. Scaling limit of the hypergeometric kernel: the Whittaker kernel

Recall that the construction of the point processes $\mathcal{P}_{z,z',\xi}$ was started from certain probability distributions on partitions of an integer number n denoted as $M_{z,z'}^{(n)}$, see §1. These distributions possess an additional important property: they converge, as $n\to\infty$, to a probability distribution on a certain limit object Ω called the *Thoma simplex*:

$$\Omega=\{\alpha_1\ge\alpha_2\ge\ldots\ge0;\ \beta_1\ge\beta_2\ge\ldots\ge0\,\Big|\,\sum_{i=1}^{\infty}(\alpha_i+\beta_i)\le1\}. \tag{5.1}$$

It is a compact topological space with respect to the topology of coordinate–wise convergence.

More precisely, for every n we embed the set $\mathbb{Y}_n$ of partitions of n into Ω by making use of the map

$$\begin{gathered}\mathbb{Y}_n\ni\lambda=(p_1,\dots,p_d\,|\,q_1,\dots,q_d)\\ \mapsto\left(\frac{p_1+1/2}{n},\dots,\frac{p_d+1/2}{n},0,0,\dots;\ \frac{q_1+1/2}{n},\dots,\frac{q_d+1/2}{n},0,0,\dots\right).\end{gathered} \tag{5.2}$$

Next, we identify $M_{z,z'}^{(n)}$ with its push–forward under the map (5.2), so that $M_{z,z'}^{(n)}$ turns into a probability measure on Ω with finite support.

Theorem 5.1. *The measures $M_{z,z'}^{(n)}$ weakly converge to a probability measure $P_{z,z'}$ on Ω as $n\to\infty$.*

Proof. This follows from a general theorem, see [KOV]. □

Recall now that to construct the process $\mathcal{P}_{z,z',\xi}$ on the lattice $\mathbb{Z}'$ we have mixed all the distributions $M_{z,z'}^{(n)}$, $n=0,1,2,\dots$, using the negative binomial distribution with suitable parameters, see §1:

$$\pi(n)=(1-\xi)^t\,\frac{(t)_n}{n!}\,\xi^n,\qquad \xi\in(0,1). \tag{5.3}$$

Let us embed $\mathbb{Z}'$ into the punctured line $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$ and then rescale the process $\mathcal{P}_{z,z',\xi}$ by multiplying the coordinates of its points by $(1-\xi)$. Then the rescaled point configuration in $\mathbb{R}^*$ that corresponds to $\lambda \in \mathbb{Y}_n$ differs from the image (5.2) of λ in Ω by the scaling factor $(1-\xi)n$.

The discrete distribution on the positive semiaxis with

$$\operatorname{Prob}\{(1-\xi)n\} = (1-\xi)^t \, \frac{(t)_n}{n!}, \quad n = 0, 1, 2, \dots$$

which depends on the parameter $\xi \in (0,1)$, converges, as $\xi \to 1$, to the gamma–distribution with parameter t

$$\gamma(ds) = \frac{s^{t-1}}{\Gamma(t)} \, e^{-s} ds. \tag{5.4}$$

This brings us to the following construction. Consider the space $\widetilde{\Omega} = \Omega \times \mathbb{R}_+$ with the probability measure

$$\widetilde{P}_{z,z'} = P_{z,z'} \otimes \frac{s^{t-1}}{\Gamma(t)} \, e^{-s} ds.$$

To any point $(\omega = (\alpha|\beta), s) \in \widetilde{\Omega}$ we associate a point configuration in $\mathbb{R}$ as follows

$$((\alpha|\beta), s) \mapsto (\alpha_1 s, \alpha_2 s, \dots; -\beta_1 s, -\beta_2 s, \dots). \tag{5.5}$$

Thus, the measure $\widetilde{P}_{z,z'}$ defines a point process on $\mathbb{R}^*$ which will be denoted as $\widetilde{\mathcal{P}}_{z,z'}$

Then the considerations above together with Theorem 5.1 suggest the following

Theorem 5.2. *The point processes $\mathcal{P}_{z,z',\xi}$ scaled by $(1-\xi)$ converge, as $\xi \to 1$, to the point process $\widetilde{\mathcal{P}}_{z,z'}$.*

We will give a rigorous formulation of this claim and its proof in our next paper. Meanwhile, we will use this theorem as a prompt.

The main result of our previous work [P.I] – [P.V] was an explicit computation of the correlation functions of $\widetilde{P}_{z,z'}$.[10] To formulate this result we shall need the classical Whittaker function $W_{\kappa,\mu}(x)$, $x > 0$.

This function can be characterized as the only solution of the Whittaker equation

$$y'' - \left(\frac{1}{4} - \frac{\kappa}{x} + \frac{\mu^2 - \frac{1}{4}}{x^2} \right) y = 0$$

such that $y \sim x^\kappa e^{-\frac{x}{2}}$ as $x \to +\infty$ (see [E1, Chapter 6]). Here κ and μ are complex parameters. Note that

$$W_{\kappa,\mu} = W_{\kappa,-\mu}.$$

We shall employ the Whittaker function for real κ and real or pure imaginary μ; then $W_{\kappa,\mu}$ is real.

We introduce the functions

$$\begin{aligned} \mathcal{P}_\pm(x) &= \frac{(zz')^{1/4}}{(\Gamma(1 \pm z)\Gamma(1 \pm z') \, x)^{1/2}} \, W_{\frac{\pm(z+z')+1}{2}, \frac{z-z'}{2}}(x), \\ \mathcal{Q}_\pm(x) &= \frac{(zz')^{3/4}}{(\Gamma(1 \pm z)\Gamma(1 \pm z') \, x)^{1/2}} \, W_{\frac{\pm(z+z')-1}{2}, \frac{z-z'}{2}}(x). \end{aligned} \tag{5.6}$$

[10] About correlation functions of point processes living on a nondiscrete space, see [DVJ], [L1], [L2].

Theorem 5.3. *The correlation functions of the process $\widetilde{\mathcal{P}}_{zz'}$ have the form*

$$\widetilde{\rho}_n^{(z,z')}(u_1,\dots,u_n) = \det\left[\mathcal{K}(u_i,u_j)\right]_{i,j=1}^n, \\ n=1,2,\dots; \quad u_1,\dots,u_n \in \mathbb{R}^*,$$

where the kernel $\mathcal{K}(u,v)$ is conveniently written in the block form

$$\mathcal{K}(u,v) = \begin{cases} \mathcal{K}_{++}(u,v), & u,v>0; \\ \mathcal{K}_{+-}(u,-v), & u>0\,, v<0; \\ \mathcal{K}_{-+}(-u,v), & u<0\,, v>0; \\ \mathcal{K}_{--}(-u,-v), & u,v<0; \end{cases}$$

with

$$\mathcal{K}_{++}(x,y) = \frac{\mathcal{P}_+(x)\mathcal{Q}_+(y) - \mathcal{Q}_+(x)\mathcal{P}_+(y)}{x-y},$$

$$\mathcal{K}_{+-}(x,y) = \frac{\mathcal{P}_+(x)\mathcal{P}_-(y) + \mathcal{Q}_+(x)\mathcal{Q}_-(y)}{x+y},$$

$$\mathcal{K}_{-+}(x,y) = -\frac{\mathcal{P}_-(x)\mathcal{P}_+(y) + \mathcal{Q}_-(x)\mathcal{Q}_+(y)}{x+y},$$

$$\mathcal{K}_{--}(x,y) = \frac{\mathcal{P}_-(x)\mathcal{Q}_-(y) - \mathcal{Q}_-(x)\mathcal{P}_-(y)}{x-y}.$$

The kernel $\mathcal{K}(u,v)$ is called the *Whittaker kernel*, see [P.IV, Th. 2.7], [BO1, Th. III].[11]

Clearly, the hypergeometric kernel (see Theorem 3.3) and the Whittaker kernel have the same structure. Theorem 5.2 prompts that the Whittaker kernel is the scaling limit of the hypergeometric one. In the next theorem we establish this fact by a direct computation.

Theorem 5.4. *For the hypergeometric kernel K given by Theorem 3.3 and the Whittaker kernel $\mathcal{K}$ given by Theorem 5.3 the following limit relation holds*

$$\lim_{\xi\nearrow 1} \frac{1}{1-\xi}\, K_{**}\left(\left[\frac{u}{1-\xi}\right], \left[\frac{v}{1-\xi}\right]\right) = \mathcal{K}_{**}(u,v), \qquad u,v\in\mathbb{R}_+\,,$$

where the subscript $**$ stands for any of the four symbols $++$, $+-$, $-+$, $--$.

Proof. Take $x,y>0$ and denote

$$k = \left[\frac{x}{1-\xi}\right], \qquad l = \left[\frac{y}{1-\xi}\right].$$

Then $(1-\xi)k \approx x$, $(1-\xi)l\approx y$. Since

$$\frac{1}{k-l} \approx \frac{1-\xi}{x-y}, \qquad \frac{1}{k+l+1} \approx \frac{1-\xi}{x+y},$$

[11] In that papers, the term ‘Whittaker kernel’ concerned the block $\mathcal{K}_{++}$ while the kernel $\mathcal{K}$ was called the matrix Whittaker kernel.

it is enough to show that

$$P_\pm(k) \approx \mathcal{P}_\pm(x), \qquad Q_\pm(k) \approx \mathcal{Q}_\pm(x). \tag{5.7}$$

We shall employ the following asymptotic relation which connects the hypergeometric function and the Whittaker function:

$$\lim_{u\to+\infty} F(a,b;u;1-\tfrac{u}{x}) = x^{\frac{a+b-1}{2}} e^{\frac{x}{2}} W_{\frac{-a-b+1}{2},\frac{a-b}{2}}(x), \qquad x>0, \tag{5.8}$$

see [E1, 6.8(1)]. Note that $\frac{\xi}{\xi-1} = 1-\frac{1}{1-\xi}$. Applying (5.8) we get the following limit relations for the hypergeometric functions entering (3.5) and (3.6):

$$F(\mp z, \mp z'; k+1; \tfrac{\xi}{\xi-1}) \approx x^{\frac{\mp(z+z')-1}{2}} e^{\frac{x}{2}} W_{\frac{\pm(z+z')+1}{2},\frac{z-z'}{2}}(x),$$
$$\frac{F(1\mp z, 1\mp z'; k+2; \frac{\xi}{\xi-1})}{(1-\xi)(k+1)} \approx x^{\frac{\mp(z+z')-1}{2}} e^{\frac{x}{2}} W_{\frac{\pm(z+z')-1}{2},\frac{z-z'}{2}}(x).$$

Next, the factor $(\psi_\pm(u))^{1/2}$ entering (3.7) behaves as follows ($\psi_\pm(u)$ was defined in (3.1)):

$$(\psi_\pm(k))^{1/2} = \left(t^{1/2}\xi^{k+1/2}\,\frac{(1\pm z)_k(1\pm z')_k}{k!k!}(1-\xi)^{\pm(z+z')}\right)^{1/2}$$
$$\approx \left(\frac{t^{1/2}e^{-x}x^{\pm(z+z')}}{\Gamma(1\pm z)\Gamma(1\pm z')}\right)^{1/2}.$$

Finally, from (3.7) we obtain

$$P_\pm(k) \approx \frac{t^{1/4}}{(\Gamma(1\pm z)\Gamma(1\pm z')x)^{1/2}} W_{\frac{\pm(z+z')+1}{2},\frac{z-z'}{2}}(x) = \mathcal{P}_\pm(x),$$
$$Q_\pm(k) \approx \frac{t^{3/4}}{(\Gamma(1\pm z)\Gamma(1\pm z')x)^{1/2}} W_{\frac{\pm(z+z')-1}{2},\frac{z-z'}{2}}(x) = \mathcal{Q}_\pm(x). \quad \square$$

Remark 5.5. As was demonstrated in §4, the "++"–block of the hypergeometric kernel turns into the Christoffel–Darboux kernel for Meixner polynomials when $z = N+\alpha$, $z' = N$, $N \in \mathbb{Z}_+$. It is well known that in the scaling limit as $\xi \to 1$, the Meixner polynomials turn into the Laguerre polynomials (see [KS], [NSU]). This agrees with the fact that for $z = N+\alpha$, $z' = N$, the restriction of the process $\widetilde{\mathcal{P}}_{z,z'}$ to the positive semiaxis coincides with the N–point Laguerre ensemble, see [P.III, Remark 2.4].

Note that the shift by N which we were doing to match $\mathcal{P}_{z,z',\xi}$ and the Meixner ensemble disappears after we take the limit.

Remark 5.6. A straightforward check shows that the scaling limit of the kernel $L(x,y)$ defined by (3.3) is the kernel $\mathcal{L}(x,y)$ of the operator $\mathcal{L} = \mathcal{K}(1-\mathcal{K})^{-1}$ where $\mathcal{K}$ is the integral operator in $L^2(\mathbb{R}^*, dx)$ corresponding to the Whittaker kernel (the kernel $\mathcal{L}(x,y)$ was explicitly computed in [P.V, Theorem 2.4]).

§6. Integrable operators

In this section we shall show that the operator given by the Whittaker kernel belongs to the class of integrable operators as defined by Its, Izergin, Korepin and Slavnov [IIKS]. We shall also argue that the hypergeometric kernel might be considered as an example of a *discrete* kernel giving an 'integrable operator'.

We shall follow [De] in our description of integrable operators.

Let Σ be an oriented contour in $\mathbb{C}$. We call an operator V acting in $L^2(\Sigma, |d\zeta|)$ *integrable* if its kernel has the form

$$V(\zeta, \zeta') = \frac{\sum_{j=1}^N f_j(\zeta) g_j(\zeta')}{\zeta - \zeta'}, \quad \zeta,\ \zeta' \in \Sigma,$$

for some functions $f_j,\ g_j,\ j = 1, \dots, N$. We shall always assume that

$$\sum_{j=1}^N f_j(\zeta) g_j(\zeta) = 0, \quad \zeta \in \Sigma,$$

so that the kernel $V(\zeta, \zeta')$ is nonsingular (this assumption is not necessary for the general theory).

The notion of an integrable operator was first introduced in [IIKS].

It turns out that for an integrable operator V the operator $R = V(1+V)^{-1}$ is also integrable.

Proposition 6.1 [IIKS]**.** *Let V be an integrable operator as described above and $R = 1 - (1+V)^{-1} = V(1+V)^{-1}$. Then the kernel $R(\zeta, \zeta')$ has the form*

$$R(\zeta, \zeta') = \frac{\sum_{j=1}^N F_j(\zeta) G_j(\zeta')}{\zeta - \zeta'}, \quad \zeta, \zeta' \in \Sigma,$$

where

$$F_j = (1+V)^{-1} f_j, \qquad G_j = (1+V^t)^{-1} g_j, \quad j = 1, \dots, N.$$

If $\sum_{j=1}^N f_j(\zeta) g_j(\zeta) = 0$ on Σ, then $\sum_{j=1}^N F_j(\zeta) G_j(\zeta) = 0$ on Σ as well.

Proof. See [KBI, ch. XIV], [De]. □

It is not difficult to show that for integrable operators V_1, V_2, the product $V_1 V_2$ is also integrable. This fact and Proposition 6.1 imply that operators of the form $I + V$ where V is integrable form a group.

A remarkable fact is that the function F_j, G_j can be expressed via a suitable Riemann–Hilbert problem, see [IIKS], [De] for details.

Now we pass to a much more special situation. Let $\Sigma = \mathbb{R}^*$. According to the splitting $\mathbb{R}^* = \mathbb{R}_+ \sqcup \mathbb{R}_-$ and further identification of $\mathbb{R}_-$ with a second copy of $\mathbb{R}_+$, we shall sometimes write the kernels of operators in $L^2(\mathbb{R}^*)$ in block form.

Consider an integral operator V on $\mathbb{R}^*$ whose kernel $V(x,y)$ has the following block form

$$[V](x,y) = \begin{bmatrix} 0 & \frac{h_+(x) h_-(y)}{x+y} \\ -\frac{h_+(y) h_-(x)}{x+y} & 0 \end{bmatrix}, \qquad x > 0, y > 0, \tag{6.1}$$

for some functions $h_+(x)$ and $h_-(x)$ defined on the positive semiaxis. Then the operator V is integrable. Indeed,

$$V(x,y)=\frac{f_1(x)g_1(y)+f_2(x)g_2(y)}{x-y}, \qquad x,y\in\mathbb{R}^*$$

where

$$f_1(x)=\begin{cases}0, & x>0\\ h_-(-x), & x<0\end{cases}, \quad f_2(x)=\begin{cases}h_+(x), & x>0\\ 0, & x<0\end{cases}$$

$$g_1(x)=\begin{cases}h_+(x), & x>0\\ 0, & x<0\end{cases}, \quad g_2(x)=\begin{cases}0, & x>0\\ h_-(-x), & x<0\end{cases}$$

Assume that there exist four functions $A_\pm(x)$, $B_\pm(x)$ defined on the positive semiaxis such that

$$\widehat{A}_\mp=\frac{B_\pm}{h_\pm^2}, \quad \widehat{B}_\mp=1-\frac{A_\pm}{h_\pm^2} \tag{6.3}$$

where

$$\widehat{\varphi}(x)=\int_{y>0}\frac{\varphi(y)dy}{x+y} \tag{6.4}$$

is the Stieltjes transform.

Then Proposition 6.1 implies that the kernel of the operator $R=V(1+V)^{-1}$ has the form

$$R(x,y)=\frac{F_1(x)G_1(y)+F_2(x)G_2(y)}{x-y}, \qquad x,y\in\mathbb{R}^*$$

with

$$F_1(x)=\begin{cases}-\frac{B_+(x)}{h_+(x)}, & x>0\\ \frac{A_-(-x)}{h_-(-x)}, & x<0\end{cases}; \qquad F_2(x)=\begin{cases}\frac{A_+(x)}{h_+(x)}, & x>0\\ \frac{B_-(-x)}{h_-(-x)}, & x<0\end{cases};$$

$$G_1(x)=\begin{cases}\frac{A_+(x)}{h_+(x)}, & x>0\\ -\frac{B_-(-x)}{h_-(-x)}, & x<0\end{cases}; \qquad G_2(x)=\begin{cases}\frac{B_+(x)}{h_+(x)}, & x>0\\ \frac{A_-(-x)}{h_-(-x)}, & x<0\end{cases}.$$

In block form the kernel $R(x,y)$ can be written as follows:

$$[R](x,y)=\begin{bmatrix}\frac{1}{h_+(x)h_+(y)}\frac{A_+(x)B_+(y)-B_+(x)A_+(y)}{x-y} & \frac{1}{h_+(x)h_-(y)}\frac{A_+(x)A_-(y)+B_+(x)B_-(y)}{x+y}\\ \frac{1}{h_-(x)h_+(y)}\frac{-A_-(x)A_+(y)-B_-(x)B_+(y)}{x+y} & \frac{1}{h_-(x)h_-(y)}\frac{A_-(x)B_-(y)-B_-(x)A_-(y)}{x-y}\end{bmatrix}$$

All these formulas work perfectly well for the Whittaker kernel. If we set, using the notation of §5,

$$h_\pm(x)=\frac{\sqrt{\sin\pi z\,\sin\pi z'}}{\pi}\,x^{\mp\frac{z+z'}{2}}e^{-x/2};$$

$$A_\pm(x)=h_\pm(x)\mathcal{P}_\pm(x), \quad B_\pm(x)=h_\pm(x)\mathcal{Q}_\pm(x),$$

then the kernel $R(x,y)$ coincides with the Whittaker kernel $\mathcal{K}(x,y)$. The form (6.1) of the kernel of $V=\mathcal{K}(1-\mathcal{K})^{-1}$ was obtained in [P.V]. The formulas (6.3) in this

case can be derived from the known formulas for the Stieltjes transform of the (suitably normalized) Whittaker function [E2, 14.3(53)].

It is a remarkable fact that all the formulas above also work for the hypergeometric kernel. This kernel lives on the lattice $\mathbb{Z}' \times \mathbb{Z}'$, so one can call the operator corresponding to the hypergeometric kernel a *discrete integrable operator*.

Indeed, exact expressions for $h_+(k)$ and $h_-(k)$ can be easily extracted from (3.3) if we take $V = L$:

$$h_\pm(k) = (\psi_\pm(k))^{1/2}.$$

Then, as before, we have, cf. (3.5)–(3.7),

$$A_\pm(k) = h_\pm(k) P_\pm(k) = R_\pm(k), \quad B_\pm(k) = h_\pm(k) Q_\pm(k) = S_\pm(k),$$

and the kernel $R(x, y)$ coincides with the hypergeometric kernel $K(x, y)$.

Relations (6.3) are exactly the relations of Lemma 3.4, see (3.14).

If we consider the continuous case, then from the general theory of Riemann–Hilbert problems one can extract the following identity for the analytic continuations of $B_\pm$, $A_\pm$:

$$A_+(\zeta) A_-(-\zeta) + B_+(\zeta) B_-(-\zeta) = h_+^2(\zeta) h_-^2(-\zeta), \quad \zeta \in \mathbb{C} \setminus \mathbb{R}. \tag{6.5}$$

It means that the determinant of the solution of the corresponding Riemann–Hilbert problem is identically equal to 1 (this follows from the fact that the determinant of the corresponding jump matrix is identically equal to 1).

Though we do not have an analog of the Riemann–Hilbert problem in the discrete case, a discrete analog of (6.5) still holds, see Lemma 3.5.

Remark 6.2. Both the Whittaker and the hypergeometric kernel possess the symmetry property

$$R(x, y) = \operatorname{sgn}(x) \operatorname{sgn}(y) R(y, x) \tag{6.6}$$

which, perhaps, emerged for the first time: most (integrable) kernels arising in Random Matrix Theory and mathematical physics are simply symmetric. The formula (6.6) means that the corresponding integral operator in $L^2(\mathbb{R}, dx)$ or in $\ell^2(\mathbb{Z}')$ is symmetric with respect to an indefinite inner product.

§7. Appendix: some relations for the Gauss hypergeometric function

In this section we prove Lemmas 3.4 and 3.5. First, we shall reformulate them using (3.5) and (3.6).

Lemma 3.4'. *For $\xi \in (0, 1)$ the following decompositions hold:*

$$\frac{F(a, b; u+1; \frac{\xi}{\xi - 1})}{u} = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k \xi^k (1-\xi)^{a+b-1}}{k!\, k!\, (u+k)} F(1-a, 1-b; k+1; \tfrac{\xi}{\xi-1}),$$

$$1 - F(a, b; u; \tfrac{\xi}{\xi-1}) = \sum_{k=0}^{\infty} \frac{(a)_{k+1} (b)_{k+1} \xi^{k+1} (1-\xi)^{a+b-1}}{k!\, k!\, (u+k)} \frac{F(1-a, 1-b; k+2; \frac{\xi}{\xi-1})}{k+1}.$$

Specifically, the series in the RHS absolutely converges for $u \neq 0, -1, -2, \dots$ and represents a meromorphic function; the both formulas are viewed as equalities of meromorphic functions in u.

Lemma 3.5'.

$$\begin{aligned}&F(-z,-z';u+1;w)F(z,z';-u;w)\\ &+zz'w(1-w)\,\frac{F(-z+1,-z'+1;u+2;w)}{u+1}\,\frac{F(z+1,z'+1;-u+1;w)}{u}\;=\;1.\end{aligned} \tag{7.1}$$

Proof of Lemma 3.4'. Let us check the first relation. The RHS has the form

$$\sum_{k=0}^{\infty}\frac{A_k}{u+k},$$

where the coefficients A_k rapidly decrease as $k\to\infty$, because of the factor ξ^k (the factor $F(1-a,1-b;k+2;\frac{\xi}{\xi-1})$ remains bounded as $k\to+\infty$, and the remaining expression has at most polynomial growth in k). Consequently, the RHS is indeed a converging series representing a meromorphic function in u. This function has simple poles at $u=0,-1,-2,\dots$. Using the formula

$$\begin{aligned}\operatorname*{Res}_{c=-k}F(a,b;c;w)&=\frac{(a)_{k+1}(b)_{k+1}(-1)^k w^{k+1}}{(k+1)!k!}F(a+k+1,b+k+1;k+2;w)\\ &=\frac{(a)_{k+1}(b)_{k+1}(-1)^k w^{k+1}(1-w)^{-a-b-k}}{(k+1)!k!}F(1-a,1-b;k+2;w)\end{aligned} \tag{7.2}$$

for the residues of the hypergeometric function, one verifies that the residues of the RHS at $u=-1,-2,\dots$ are the same as for the LHS, and it is directly seen that the residues at $u=0$ coincide, too. Moreover, the same claim holds not only for $\xi\in(0,1)$ but for any complex ξ ranging over the unit disc $|\xi|<1$, and the both sides are holomorphic in ξ.

Let us expand both the LHS and the RHS into Taylor series in ξ and compare the respective Taylor coefficients. Each Taylor coefficient (on the left and on the right), viewed as a function in a,b,u, is a rational expression which is polynomial in a,b. This implies that it suffices to prove our relation, say, for $a=0,-1,-2,\dots$.

Thus, we may assume that $a=0,-1,-2,\dots$. For these values of a, the LHS becomes a rational function in u, and the series in the RHS terminates and, consequently, is a rational function in u, too. Next, we know that the both sides have the same singularities. Finally, they both behave as $O(1/u)$ as $|u|\to\infty$ (indeed, as was mentioned above, the hypergeometric function in the numerator of the LHS is $1+O(1/u)$, so that the whole expression is $O(1/u)$, and for the RHS the same holds, because the series terminates). Consequently, the both sides are identical. This concludes the proof of our first relation.

The second relation is verified similarly. $\square$

Proof of Lemma 3.5'. We use the same argument as in the previous proof. The desired relation can be viewed as an equality of power series in the variable w. The coefficients of the series are polynomials in z,z', so that we may assume, without loss of generality, that one of the parameters z,z' takes positive integer values while another takes negative integer values. In this case all the four hypergeometric series entering our relation terminate and, so, are rational functions in u.

Next, we examine the possible singularities of the LHS of (7.1). Here only simple poles at the points $u\in\mathbb{Z}$ may occur, but it turns out that the residue at any $u\in\mathbb{Z}$

vanishes. That is, the contributions of the two products cancel each other. To see this, one can use, for example, (7.2).

Finally, we remark that, under our specialization of z, z', the LHS of (7.1) is $1 + O(1/u)$ as $|u| \to \infty$. This concludes the proof. $\square$

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